[1]

EXPLICIT SOLUTION OF THE PROBLEM OF EQUIVALENCE FOR SOME PAINLEVE EQUATIONS

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Abstact. For an arbitrary ordinary second order differential equation a test is constructed that checks if this equation is equivalent to Painleve I, II or Painleve III with three zero parameters equations under the substitutions of variables. If it is true then in case the Painleve equations I and II an explicite change of variables is given that is written using the differential invariants of the equation.

Keywords: Painleve equations, equivalence problem, differential invariants.

1. Introduction

It is well-known fact that the ordinary differential equations of the form

$$y''=P(x,y)+3\,Q(x,y)y'+3\,R(x,y)y'^2+S(x,y)y'^3 \tag{1}$$

preserve their form under the action of arbitrary point transformations

$$\tilde{x}=\tilde{x}(x,y),\quad \tilde{y}=\tilde{y}(x,y). \tag{2}$$

So we can use of geometric methods to study these equations. See [1], [2], [3], [4], [5]. In particular, we can build differential invariants associated with the equation (1) from its coefficients - functions $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$, and their derivatives. Such invariants are called *Cartan invariants*. See [6], [7], [8], [9].

All six famous Painleve equations ([10], [11], [12]) have the form (1). The first, second and third Painleve equations respectively are:

$$y''=6y^2+x,\qquad y''=2y^3+xy+a,\qquad y''=\frac{1}{y}(y')^2-\frac{1}{x}y'+\frac{1}{x}(ay^2+b)+cy^3+\frac{d}{y}.$$

We use methods of differential invariants, described in the papers [13], [14], [15] in order to solve the problem of the equivalence for Painleve I and II equations and for Painleve III equation with three zero parameters.

Similar studies were conducted previously, see [16], [17], [18], [19], [20]. However, for the first time the check test for the equivalence is formulated in the effectively verifiable manner, it can be programmed. This is a continuation of work [21].

For further calculations we also need to define pseudotensorial field and its covariant derivative (as they were formulated in [13]).

Definition 1. *The pseudotensorial field weights m of valence (r,s) is called indexed set of variables that transform with the following rule under change of coordinate system*

$$F^{i_1\ldots i_r}_{j_1\ldots j_s}=(\det T)^m\sum_{p_1\ldots p_r}\sum_{q_1\ldots q_s}S^{i_1}_{p_1}\ldots S^{i_r}_{p_r}T^{q_1}_{j_1}\ldots T^{q_s}_{j_s}\tilde{F}^{p_1\ldots p_r}_{q_1\ldots q_s},$$

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here T is the inverse transfer matrix when one changes one coordinate system to another system in the plane. We see that there is the only factor $(\,detT)^m$ that distinguishes Definition 1 from the classical definition of tensorial field.

Definition 2. *The following object is called the covariant derivative of pseudotensorial field F of valence (r,s) and weight m:*

$$\nabla_k F^{i_1\dots i_r}_{j_1\dots j_s} = \frac{\partial F^{i_1\dots i_r}_{j_1\dots j_s}}{\partial x^k} + \sum_{n=1}^{r}\sum_{v_n=1}^{2}\Gamma^{i_n}_{kv_n}F^{i_1\dots v_n\dots i_r}_{j_1\dots j_s} - \\ - \sum_{n=1}^{s}\sum_{w_n=1}^{2}\Gamma^{w_n}_{kj_n}F^{i_1\dots i_r}_{j_1\dots w_n\dots j_s} + m\varphi_k F^{i_1\dots i_r}_{j_1\dots j_s}$$

Under the covariant differentiation the pseudotensorial field F of valence (r,s) and weight m becomes the pseudotensorial field of valence $(r,s+1)$ and weight m.

Only the term $m\varphi_k F^(i_1\dots i_r)_(j_1\dots j_s)$ distinguishes Definition 2 from the definition of covariant derivative of tensor fields. Here φ_1 and φ_2 are auxiliary coefficients, the explicit formula (25), (26) for their calculation are contained in the Appendix.

2. Painleve I equation

Suppose we have a certain equation (1). We are looking for a change of variables, which translates it to the Painleve I equation

$$\tilde{y}'' = 6\tilde{y}^2 + \tilde{x}. \tag{3}$$

There are two pseudovectorial fields α of weight 2 and θ of weight -1 associated with equation (3) (for details, see [14]). Explicit formulas (16), (24) for calculating the coordinates of these fields are contained in the Appendix.

For the equation (3) they are given by:

$$\tilde{\alpha}^1 = \tilde{B} = 0, \quad \tilde{\alpha}^2 = -\tilde{A} = -12, \quad \tilde{\theta}^1 = -\frac{1}{12}, \quad \tilde{\theta}^2 = 0.$$

Then the transformation laws of these fields under the change of coordinates (2) are the following:

$$\begin{aligned} \begin{pmatrix} B \\ -A \end{pmatrix} &= \det T \begin{pmatrix} \tilde{y}_{0.1} & -\tilde{x}_{0.1} \\ -\tilde{y}_{1.0} & \tilde{x}_{1.0} \end{pmatrix} \begin{pmatrix} 0 \\ 12 \end{pmatrix}, \\ \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} &= \frac{1}{\det^2 T} \begin{pmatrix} \tilde{y}_{0.1} & -\tilde{x}_{0.1} \\ -\tilde{y}_{1.0} & \tilde{x}_{1.0} \end{pmatrix} \begin{pmatrix} -1/12 \\ 0 \end{pmatrix}. \end{aligned} \tag{4}$$

In addition, the coefficients of the equation (3) define the pseudoinvariants, explicit formulas for the calculation of which are contained in the Appendix. N of weight 2 (18); Ω of weight 1 (21), (22); Θ of weight -2 (23); L of weight -4 (27); L_1 weight -5 (28); V of weight -3 (30); W of weight -6 (29). Now we can construct the set of invariants:

$$I_1 = \frac{L_1^4}{L^5}, \qquad I_2 = \frac{\Theta^2}{L}.$$

In this case pseudoinvarians are:

$$\tilde{N} = 0, \quad \tilde{\Omega} = 0, \quad \tilde{\Theta} = -\frac{\tilde{y}}{12}, \quad \tilde{L} = \frac{\tilde{x}}{12^3}, \quad \tilde{L}_1 = -\frac{1}{12^4}, \quad \tilde{V} = 0, \quad \tilde{W} = 0.$$

We see that pseudoinvariant $\tilde{L}_1$ of weight -5 for the Painleve equation I (3) is a constant.

We can calculate the determinant of the inverse matrix of transition $\det T$ using the transformation law for the $\tilde{L}_1$ under the change of coordinates:

$$L_1 = \frac{\tilde{L}_1}{(\det T)^5} = \frac{-1}{12^4(\det T)^5}, \quad \det T = \sqrt[5]{-\frac{1}{12^4 L_1}}.$$

Now from (4) we calculate partial derivatives of the coordinate functions:

$$\tilde{x}_{1.0} = -A\sqrt[5]{-\frac{L_1}{12}}, \quad \tilde{x}_{0.1} = -B\sqrt[5]{-\frac{L_1}{12}}, \quad \tilde{y}_{1.0} = \frac{\theta^2}{\sqrt[5]{12^3L_1^2}}, \quad \tilde{y}_{0.1} = -\frac{\theta^1}{\sqrt[5]{12^3L_1^2}}.$$

For any equation (1) functions A, B, θ^1, θ^2, L_1 are known, they are determined by functions P, Q, R, S via explicite formulae (16), (24), (28) given in the Appendix. From the last formula it is easy to obtain conditions for the compatibility:

$$\left(A\sqrt[5]{-\frac{L_1}{12}}\right)_y = \left(B\sqrt[5]{-\frac{L_1}{12}}\right)_x, \quad \left(\frac{\theta^2}{\sqrt[5]{12^3L_1^2}}\right)_y = \left(-\frac{\theta^1}{\sqrt[5]{12^3L_1^2}}\right)_x.$$

The first condition of compatibility gives us the following expression

$$5L_1(A_y - B_x) + A(L_1)_y - B(L_1)_x = 5L_1(A_y - B_x) - \nabla_\alpha L_1 + 5L_1(\varphi_1 B - \varphi_2 A) =$$
$$= 5L_1(A_y - B_x - \varphi_1 B + \varphi_2 A) - V = 6L_1 N - V = 0.$$

We have used the definition of pseudoinvariant V

$$V = \nabla_\alpha L_1 = (L_1)_x B - (L_1)_y A - 5L_1(B\varphi_1 - A\varphi_2),$$

as well as the following clearly verifiable identity:

$$B_x - A_y = -\frac{6}{5}N + \varphi_2 A - \varphi_1 B. \tag{5}$$

From the second condition of compatibility:

$$\frac{5}{2}L_1\left(\theta^1_x + \theta^2_y\right) - \left((L_1)_x\theta^1 + (L_1)_y\theta^2\right) =$$
$$= \frac{5}{2}L_1\left(\theta^1_x + \theta^2_y\right) - \nabla_\theta L_1 - 5L_1(\varphi_1\theta^1 + \varphi_2\theta^2) =$$
$$= \frac{5}{2}L_1\left((\Theta_y - 2\varphi_2\Theta)_x + (-\Theta_x + 2\varphi_1\Theta)_y\right) - W-$$
$$- 5L_1\left(\varphi_1(\Theta_y - 2\varphi_2\Theta) + \varphi_2(-\Theta_x + 2\varphi_1\Theta)\right) =$$
$$= \frac{5}{2}L_1\left((\Theta_{xy} - 2(\varphi_2)_x\Theta - 2\varphi_2\Theta_x) + (-\Theta_{xy} + 2(\varphi_1)_y\Theta + 2\varphi_1\Theta_y)\right) - W-$$
$$- 5L_1\left(\varphi_1\Theta_y - \varphi_2\Theta_x)\right) = 5L_1((\varphi_1)_y - (\varphi_2)_x)\Theta - W = 3L_1\Omega\Theta - W = 0.$$

We have used the definition of pseudoinvariants W and Ω, pseudovectorial field θ:

$$W = \nabla_\theta L_1 = (L_1)_x\theta^1 + (L_1)_y\theta^2 - 5L_1(\varphi_1\theta^1 + \varphi_2\theta^2), \quad \Omega = \frac{5}{3}\left((\varphi_1)_y - (\varphi_2)_x\right),$$
$$\theta^1 = \Theta_y - 2\varphi_2\Theta, \qquad \theta^2 = -\Theta_x + 2\varphi_1\Theta.$$

As for the Painleve I equation all pseudoinvariants $\tilde{N}$, $\tilde{\Omega}$, $\tilde{V}$, $\tilde{W}$ are identically equal to zero, they must be zero for any equation that is equivalent to Painleve I equation.

Since $N = 0$ and $V = 0$, the first condition of compatibility is true, and since $W = 0$ and $\Omega = 0$, the second condition of compatibility is true. Thus, realisation of these conditions is equal to the existing of the point substitution of variables, which reduces original equation (1) to equation (3).

Invariants of the equation (3) are given by:

$$I_1 = \frac{1}{12\tilde{x}^5}, \quad I_2 = \frac{12\tilde{y}^2}{\tilde{x}}.$$

Let us resolve I_1 and I_2 relatively $\tilde{x}$ and $\tilde{y}$. The explicite change of variables is given by (6) and find the explicite change of variables:

$$\tilde{x} = \frac{1}{\sqrt[5]{12I_1}}, \qquad \tilde{y} = \pm\frac{\sqrt{I_2}}{\sqrt[5]{12^3}\,\sqrt[10]{I_1}}. \tag{6}$$

Theorem 1 *Equation (1) is equaivalent to Painleve I equation (3) under transformations (2) if and only if the following conditions are true: 1) $F = 0$ (17), but $A \neq 0$ or $B \neq 0$ (16), 2) $\Omega = 0$ (21), (22), 3) $N = 0$ (18), 4) $W = 0$ (29), 5) $V = 0$ (30), 6) $\Theta \neq 0$ (23), 7) $L_1 \neq 0$ (28). The explicite change of variables is given by (6).*

Example 1. The following equation is equivalent to the Painleve I equation:

$$y'' = -\sin^3 y(6x\cos^2 y + \sin y) + \frac{1}{x}(-18x^3\cos^3 y\sin^2 y - 3x^2\sin^3 y\cos y - 2)y' -$$
$$-(18x^3\cos^4 y\sin y + 3x^2\sin^2 y\cos^2 y)y'^2 - (6x^4\cos^5 y + x^3\sin y\cos^3 y + x)y'^3.$$

Invariants and the explicite change of variables are the following:

$$I_1 = \frac{1}{12}\frac{1}{x^5\sin^5 y}, \quad I_2 = \frac{12x\cos^2 y}{\sin y}, \quad \tilde{y} = x\cos y, \quad \tilde{x} = x\sin y.$$

3. Painleve II equation

For a certain equation of type (1) we are looking for a change of variables that transformes it into the Painleve II equation

$$\tilde{y}'' = 2\tilde{y}^3 + \tilde{x}\tilde{y} + a. \tag{7}$$

There are two pseudovectorial fields α of weight 2 (16) and ξ of weight 3 (31) associated with equation (7). Explicit formulas for their calculation are contained in Appendix. For equation (7) they are:

$$\tilde{\alpha}^1 = \tilde{B} = 0, \quad \tilde{\alpha}^2 = -\tilde{A} = -12\tilde{y}, \quad \tilde{\xi}^1 = -\frac{24}{5\tilde{y}}, \quad \tilde{\xi}^2 = 0.$$

Under the change of variables they transform into the rule:

$$\begin{pmatrix} B \\ -A \end{pmatrix} = (\det T)\begin{pmatrix} \tilde{y}_{0.1} & -\tilde{x}_{0.1} \\ -\tilde{y}_{1.0} & \tilde{x}_{1.0} \end{pmatrix}\begin{pmatrix} 0 \\ -12\tilde{y} \end{pmatrix},$$

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = (\det T)^2\begin{pmatrix} \tilde{y}_{0.1} & -\tilde{x}_{0.1} \\ -\tilde{y}_{1.0} & \tilde{x}_{1.0} \end{pmatrix}\begin{pmatrix} -24/(5\tilde{y}) \\ 0 \end{pmatrix}.$$

Pseudoinvariants M of weight 4 (19), (20), N of weight 2 (18), Ω of weight 1 (21), (22) and Γ (32) for the equation (7) are given by:

$$\tilde{M} = \frac{288}{5}, \qquad \tilde{N} = 4, \qquad \tilde{\Omega} = 0, \qquad \tilde{\Gamma} = \frac{48}{25}\frac{2\tilde{y}^3 + \tilde{x}\tilde{y} + a}{\tilde{y}^3}.$$

Invariants of the equation are calculated by the formulas:

$$
\begin{aligned}
&I_1 = \frac{M}{N^2}, \quad I_3 = \frac{\Gamma}{M}, \quad I_6 = \frac{\nabla_\alpha I_3}{N} = \frac{B(I_3)'_x - A(I_3)'_y}{N},\\
&I_9 = \frac{(\nabla_\gamma I_3)^2}{N^3} = \frac{(\xi^1 (I_3)'_x + \xi^2 (I_3)'_y)^2}{N^3}.
\end{aligned} \tag{8}
$$

Similarly to the previously discussed case of the conversion formula for N let us find $\det T$:

$$N = 4(\det T)^2, \qquad \det T = \frac{\sqrt{N}}{2},$$

then

$$\frac{\tilde{y}_{0.1}}{\tilde{y}} = -\frac{5}{6}\frac{\xi^1}{N}, \quad \frac{\tilde{y}_{1.0}}{\tilde{y}} = \frac{5}{6}\frac{\xi^2}{N} \tag{9}$$

and the corresponding compatibility condition has the form:

$$\left(-\frac{\xi^1}{N}\right)_x = \left(\frac{\xi^2}{N}\right)_y.$$

It is equivalent to

$$
\begin{aligned}
&N(\xi^1_x + \xi^2_y) - (\xi^1 N_x + \xi^2 N_y) = N(N_y + 2\varphi_2 N)_x + N(-N_x - 2\varphi_1 N)_y -\\
&- N_x(N_y + 2\varphi_2 N) - N_y(-N_x - 2\varphi_1 N) = 2N^2((\varphi_2)_x - (\varphi_1)_y) = \frac{10}{3} N^2 \Omega = 0.
\end{aligned}
$$

We have used the definition

$$\xi^1 = N_y + 2\varphi_2 N, \qquad \xi^2 = -N_x - 2\varphi_1 N. \tag{10}$$

This condition is fulfilled if $\Omega = 0$ As we prove the existence of $\tilde{y}$, let's substitute it into the first equality:

$$\tilde{x}_{0.1} = \frac{B}{6\tilde{y}\sqrt{N}}, \quad \tilde{x}_{1.0} = \frac{A}{6\tilde{y}\sqrt{N}}, \quad \left(\frac{B}{\tilde{y}\sqrt{N}}\right)_x = \left(\frac{A}{\tilde{y}\sqrt{N}}\right)_y.$$

Let us write this in more details:

$$
\begin{aligned}
&\frac{B_x - A_y}{\tilde{y}\sqrt{N}} + \frac{A\tilde{y}_{0.1} - B\tilde{y}_{1.0}}{\tilde{y}^2\sqrt{N}} + \frac{AN_y - BN_x}{2\tilde{y}N\sqrt{N}} =\\
&= \frac{-\frac{6}{5}N + \varphi_2 A - \varphi_1 B}{\tilde{y}\sqrt{N}} - \frac{5(B\xi^2 + A\xi^1)}{6\tilde{y}N\sqrt{N}} + \frac{A(\xi^1 - 2\varphi_2 N) + B(\xi^2 + 2\varphi_1 N)}{2\tilde{y}N\sqrt{N}} =\\
&= \frac{-6N}{5\tilde{y}\sqrt{N}} - \frac{B\xi^2 + A\xi^1}{3\tilde{y}N\sqrt{N}} = \frac{-18N^2 + 5M}{15\tilde{y}N\sqrt{N}} = \frac{-18 + 5I_1}{15\tilde{y}N^3\sqrt{N}} = 0.
\end{aligned}
$$

The first condition of compatibility is satisfied if $I_1 = 18/5$. We used formula (5), (9), (10) and the definition

$$M = -A\xi^1 - B\xi^2.$$

Values of basic invariants (8) to the equation (7):

$$I_1 = \frac{18}{5}, \quad I_3 = \frac{2\tilde{y}^3 + \tilde{x}\tilde{y} + a}{30\tilde{y}^3}, \quad I_6 = \frac{2\tilde{x}\tilde{y} + 3a}{10\tilde{y}^3}, \quad I_9 = \frac{1}{2500\tilde{y}^6}.$$

Let us construct a new invariant, which is up to the sign equals the parameter of equation (7):

$$J = \frac{1}{50}\frac{4 + 10I_6 - 60I_3}{\sqrt{I_9}} = \pm a. \tag{11}$$

Lemma 1. *Equations Painleve II with the different parameters $a_1 \neq \pm a_2$ are non-equivalent.*

Via the formula of the invarians we find the explicite change of variables:

$$\tilde{y} = \frac{1}{\sqrt[6]{2500 I_9}}, \qquad \tilde{x} = \frac{5I_6}{\sqrt[6]{2500 I_9}} - \frac{3}{2} J \sqrt[6]{2500 I_9}. \tag{12}$$

Theorem 2. *An arbitrary equation (1) is equivalent to the Painleve II equation with parameter $a = \pm J$ (11) if and only if the following conditions are true: 1) $F = 0$ (17), but $A \neq 0$ or $B \neq 0$ (16), 2) $\Omega = 0$ (21), (22) , 3) $M \neq 0$ (19), (20), 4) $I_1 = 18/5$ (8). The explicit change of variables is (12).*

Example 2. Under a linear change of variables equation 6.9 from [22] is reduced to the Painleve II equation with parameter $\pm J$:

$$y'' = -ay^3 - bxy - cy - d.$$

$$J = -\sqrt{\frac{a}{2}} \cdot \frac{d}{b}, \qquad \tilde{y} = \sqrt{\frac{a}{2}} \cdot \frac{y}{\sqrt[3]{b}}, \qquad \tilde{x} = -\frac{bx + c}{\sqrt[3]{b^2}}.$$

4. Equation Painleve III with three zero parametrs

A general form of the Painleve equations III is the following:

$$y'' = \frac{1}{y}(y')^2 - \frac{1}{x}y' + \frac{1}{x}(ay^2 + b) + cy^3 + \frac{d}{y}.$$

It is a 4-parameter family of equations, which we denote by $PIII(a, b, c, d)$.

If three out of four of these parameters are zero, then these equations of Painleve III have special properties:

1. They have a two-dimensional algebra of point symmetries and hence integrable. See [12].
2. All these equations are equivalent to each other. Referring to work [19], we write the change of variables:

$$PIII(0, b, 0, 0) \xrightarrow{1)} PIII(-b, 0, 0, 0) \xrightarrow{2)} PIII(0, 0, -b, 0) \xrightarrow{3)} PIII(0, 0, 0, b),$$

here $1), 3)$ $x = \tilde{x}$, $y = 1/\tilde{y}$, $\quad 2)$ $x = \tilde{x}^2/2$, $y = \tilde{y}^2$.

Therefore it makes sense to solve the problem of equivalence for the same type of equations. We chose $PIII(0, b, 0, 0)$:

$$y'' = \frac{1}{y}(y')^2 - \frac{1}{x}y' + \frac{b}{x}. \tag{13}$$

For the equation (13) the coordinates of the pseudovectorial fields α of weight 2 (16) and ξ of the weights 3 (31) are:

$$\tilde{A} = \frac{b}{\tilde{x}\tilde{y}^3}, \qquad \tilde{B} = 0, \qquad \tilde{\xi}^1 = -\frac{1}{15}\frac{b}{\tilde{x}\tilde{y}^4}, \qquad \tilde{\xi}^2 = -\frac{1}{15}\frac{b}{\tilde{x}^2\tilde{y}^3},$$

and values of pseudoinvarians M of the weight 4 (19), (20), N of weight 2 (18) and Ω of weight 1 (21) , (22) are:

$$\tilde{N} = -\frac{1}{3}\frac{b}{\tilde{x}\tilde{y}^3}, \qquad \tilde{M} = \frac{1}{15}\frac{b^2}{\tilde{x}^2\tilde{y}^6}, \qquad \tilde{\Omega} = 0.$$

The basic invariants of the equation are:

$$I_1 = \frac{M}{N^2} = \frac{3}{5}, \qquad I_2 = \frac{\Omega^2}{N} = 0, \qquad I_3 = \frac{\Gamma}{M} = \frac{1}{15}.$$

Let us calculate $\det T$ from the transformation law of the pseudoinvariant N:

$$\det T = \frac{\sqrt{-3N\tilde{x}\tilde{y}^3}}{\sqrt{b}},$$

then from the laws of transformation of the pseudovectorials fields α and ξ we get

$$\tilde{x}_{0.1} = \frac{B\sqrt{\tilde{x}\tilde{y}}}{\sqrt{-3bN}}, \qquad \tilde{x}_{1.0} = \frac{A\sqrt{\tilde{x}\tilde{y}}}{\sqrt{-3bN}}, \tag{14}$$

$$\tilde{y}_{0.1} = \frac{5\xi^1\tilde{y}}{N} + \frac{B\tilde{y}^2}{\sqrt{-3bN\tilde{x}\tilde{y}}}, \qquad \tilde{y}_{1.0} = -\frac{5\xi^2\tilde{y}}{N} + \frac{A\tilde{y}^2}{\sqrt{-3bN\tilde{x}\tilde{y}}}. \tag{15}$$

In this case the compatibility conditions are the following:

$$\left(\frac{B\sqrt{\tilde{y}}}{\sqrt{N}}\right)_x = \left(\frac{A\sqrt{\tilde{y}}}{\sqrt{N}}\right)_y, \quad \left(\frac{5\xi^1}{N} + \frac{B\sqrt{\tilde{y}}}{\sqrt{-3bN\tilde{x}}}\right)_x = \left(-\frac{5\xi^2}{N} + \frac{A\sqrt{\tilde{y}}}{\sqrt{-3bN\tilde{x}}}\right)_y$$

Let us write the first condition of compatibility, using formulas (14), (15):

$$\frac{(B_{1.0} - A_{0.1})\sqrt{\tilde{y}}}{\sqrt{N}} + \frac{(B\tilde{y}_{1.0} - A\tilde{y}_{0.1})}{2\sqrt{N}\sqrt{\tilde{y}}} - \frac{1}{2}\frac{\sqrt{\tilde{y}}(BN_{1.0} - AN_{0.1})}{\sqrt{N^3}} =$$
$$= -\frac{6}{5}\sqrt{N}\sqrt{\tilde{y}} - \frac{1}{2}\frac{\sqrt{\tilde{y}}M}{\sqrt{N^3}} - \frac{5\sqrt{\tilde{y}}}{2\sqrt{N}}(B\xi^2 + A\xi^1) = \sqrt{\tilde{y}}\sqrt{N}\left(-\frac{6}{5} - \frac{1}{2}\frac{M}{N^2} + \frac{5}{2}\frac{M}{N^2}\right) =$$
$$= \sqrt{\tilde{y}}\sqrt{N}\left(-\frac{6}{5} + 2I_1\right) = 0.$$

It is true if $I_1 = 3/5$. The second condition can be written as follows:

$$\frac{5(\xi^1_x + \xi^2_y)}{N} - \frac{5(\xi^1 N_{1.0} + \xi^2 N_{0.1})}{N^2} + \frac{(B_{1.0} - A_{0.1})\sqrt{\tilde{y}}}{\sqrt{-3bN\tilde{x}}} -$$
$$- \frac{\sqrt{\tilde{y}}(BN_{1.0} - AN_{0.1})}{2\sqrt{-3b\tilde{x}N^3}} + \frac{(B\tilde{y}_{1.0} - A\tilde{y}_{0.1})}{2\sqrt{-3bN\tilde{x}\tilde{y}}} - \frac{\sqrt{\tilde{y}}(B\tilde{x}_{1.0} - A\tilde{x}_{0.1})}{2\sqrt{-3bN\tilde{x}^3}} =$$
$$= \frac{50}{3}\Omega + \frac{\sqrt{\tilde{y}}}{\sqrt{-3bN\tilde{x}}}\left(-\frac{6}{5}N - \frac{M}{2N} + \frac{5M}{2N}\right) = \frac{50}{3}\Omega + \frac{\sqrt{\tilde{y}}N}{\sqrt{-3bN\tilde{x}}}\left(-\frac{6}{5} + 2I_1\right) = 0.$$

The second condition true if $\Omega = 0$.

Theorem 3. *An arbitrary equation (1) is equivalent to the Painleve III equation with three zero parameters if and only if the following conditions are true: 1) $F = 0$ (17), but $A \neq 0$ or $B \neq 0$ (16), 2) $\Omega = 0$ (21), (22) , 3) $M \neq 0$ (19), (20), 4) $I_1 = 3/5$.*

In this case, we can not write an explicit change of variables via invariants of the equation because they are constants.

5. Conclusion

Thus we have found the explicite verification test for a second order ODE to be equivalent to Painleve equations I, II and III with three zero parameters. For the first two cases a point transformation of variables is found, which is written using the differential invariants of the equation.

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7. Appendix

Let us denote $K_{i.j} = \partial^{i+j}K/\partial x^i \partial y^j$.

The coordinates of the pseudovectorial field α are $\alpha^1 = B$, $\alpha^2 = -A$, where

(16)

$$A = P_{0.2} - 2Q_{1.1} + R_{2.0} + 2PS_{1.0} + SP_{1.0} - 3PR_{0.1} - 3RP_{0.1} - 3QR_{1.0} + 6QQ_{0.1},$$
$$B = S_{2.0} - 2R_{1.1} + Q_{0.2} - 2SP_{0.1} - PS_{0.1} + 3SQ_{1.0} + 3QS_{1.0} + 3RQ_{0.1} - 6RR_{1.0}.$$

The first pseudoinvariant F of the weight 5 is:

$$3F^5 = AG + BH, \qquad \text{where} \tag{17}$$

$$G = -BB_{1.0} - 3AB_{0.1} + 4BA_{0.1} + 3SA^2 - 6RBA + 3QB^2,$$
$$H = -AA_{0.1} - 3BA_{1.0} + 4AB_{1.0} - 3PB^2 + 6QAB - 3RA^2.$$

The pseudoinvariant N of the weight 2 in cases $A \neq 0$ and $B \neq 0$ respectively is:

$$N = -\frac{H}{3A}, \qquad N = \frac{G}{3B}. \tag{18}$$

The pseudoinvariant M of the weight 4 in the case $A \neq 0$ is:

(19)

$$M = -\frac{12BN(BP + A_{1.0})}{5A} + BN_{1.0} + \frac{24}{5}BNQ + \frac{6}{5}NB_{1.0} + \frac{6}{5}NA_{0.1} - AN_{0.1} - \frac{12}{5}ANR$$

and in the case $B \neq 0$ is:

(20)

$$M = -\frac{12AN(AS - B_{0.1})}{5B} - AN_{0.1} + \frac{24}{5}ANR - \frac{6}{5}NA_{0.1} - \frac{6}{5}NB_{1.0} + BN_{1.0} - \frac{12}{5}BNQ.$$

The pseudoinvariant Ω of the weight 1 in the case $A \neq 0$ is:

(21)

$$\Omega = \frac{2BA_{1.0}(BP + A_{1.0})}{A^3} - \frac{(2B_{1.0} + 3BQ)A_{1.0}}{A^2} + \frac{(A_{0.1} - 2B_{1.0})BP}{A^2} -$$
$$- \frac{BA_{2.0} + B^2P_{1.0}}{A^2} + \frac{B_{2.0}}{A} + \frac{3B_{1.0}Q + 3BQ_{1.0} - B_{0.1}P - BP_{0.1}}{A} + Q_{0.1} - 2R_{1.0}$$

and in the case $B \neq 0$ is:

(22)

$$\Omega = \frac{2AB_{0.1}(AS - B_{0.1})}{B^3} - \frac{(2A_{0.1} - 3AR)B_{0.1}}{B^2} + \frac{(B_{1.0} - 2A_{0.1})AS}{B^2} +$$
$$+ \frac{AB_{0.2} - A^2S_{0.1}}{B^2} - \frac{A_{0.2}}{B} + \frac{3A_{0.1}R + 3AR_{0.1} - A_{1.0}S - AS_{1.0}}{B} + R_{1.0} - 2Q_{0.1}.$$

The pseudocovectorial field ω of the weight -1 in the case $A \neq 0$ is:

$$\begin{aligned}\omega_1 =&\frac{12PR}{5A} - \frac{54}{25}\frac{Q^2}{A} - \frac{P_{0.1}}{A} + \frac{6Q_{1.0}}{5A} - \frac{PA_{0.1} + BP_{1.0} + A_{2.0}}{5A^2} - \frac{2B_{1.0}P}{5A^2} + \\ &+ \frac{3QA_{1.0} - 12PBQ}{25A^2} + \frac{6B^2P^2 + 12BPA_{1.0} + 6A_{1.0}^2}{25A^3},\\ \omega_2 =&\frac{-5BP_{0.1} + 6BQ_{0.1} + 12RBP}{5A^2} - \frac{54}{25}\frac{BQ^2}{A^2} - \frac{2BB_{1.0}P + BA_{0.1}P + B^2P_{1.0} + BA_{2.0}}{5A^3} - \\ &- \frac{12B^2PQ}{25A^3} + \frac{3BQA_{1.0}}{25A^3} + \frac{6BA_{1.0}^2 + 6B^3P^2 + 12B^2A_{1.0}P}{25A^4}\end{aligned}$$

and in the case $B \neq 0$ is:

$$\begin{aligned}\omega_1 =&\frac{5AS_{1.0} - 6AR_{0.1} + 12QAS}{5B^2} - \frac{54}{25}\frac{AR^2}{B^2} + \frac{2AA_{0.1}S + AB_{1.0}S + A^2S_{0.1} - AB_{0.2}}{5B^3} - \\ &- \frac{12A^2SR}{25B^3} + \frac{3ARB_{0.1}}{25B^3} + \frac{6AB_{0.1}^2 + 6A^3S^2 - 12A^2B_{0.1}S}{25B^4},\\ \omega_2 =&\frac{12SQ}{5B} - \frac{54}{25}\frac{R^2}{B} + \frac{S_{1.0}}{B} - \frac{6R_{0.1}}{5B} + \frac{SB_{1.0} + AS_{0.1} - B_{0.2}}{5B^2} + \frac{2A_{0.1}S}{5B^2} - \\ &- \frac{3RB_{0.1} + 12SAR}{25B^2} + \frac{6A^2S^2 - 12B_{0.1}AS + 6B_{0.1}^2}{25B^3}.\end{aligned}$$

The pseudoinvariant Θ of the weight -2 is given by:

$$\Theta = \frac{\omega_1}{A}, \qquad \Theta = \frac{\omega_2}{B}. \tag{23}$$

The pseudovectorial field θ of the weight -1 is:

$$\theta^1 = \Theta_{0.1} - 2\varphi_2\Theta, \qquad \theta^2 = -\Theta_{1.0} + 2\varphi_1\Theta, \tag{24}$$

where φ_i in the case $A \neq 0$ are:

$$\varphi_1 = -3\frac{BP + A_{1.0}}{5A} + \frac{3}{5}Q, \quad \varphi_2 = 3B\frac{BP + A_{1.0}}{5A^2} - 3\frac{B_{1.0} + A_{0.1} + 3BQ}{5A} + \frac{6}{5}R, \tag{25}$$

and in the case $B \neq 0$ are:

$$\varphi_1 = -3A\frac{AS - B_{0.1}}{5B^2} - 3\frac{A_{0.1} + B_{1.0} - 3AR}{5B} - \frac{6}{5}Q, \quad \varphi_2 = 3\frac{AS - B_{0.1}}{5B} - \frac{3}{5}R. \tag{26}$$

The pseudoinvariant L of the weight -4 is:

$$\begin{aligned}L =&\theta^1\theta^2(\theta^1_{1.0} - \theta^2_{0.1}) + (\theta^2)^2\theta^1_{0.1} - (\theta^1)^2\theta^2_{1.0} - \\ &- P(\theta^1)^3 - 3Q(\theta^1)^2\theta^2 - 3R\theta^1(\theta^2)^2 - S(\theta^2)^3 - \frac{1}{2}\Theta^2.\end{aligned} \tag{27}$$

The pseudoinvariant L_1 of the weight -5 is:

$$L_1 = L_{1.0}\theta^1 + L_{0.1}\theta^2 - 4L(\varphi_1\theta^1 + \varphi_2\theta^2). \tag{28}$$

The pseudoinvariant W of the weight -6 is:

$$W = \nabla_\theta L_1 = (L_1)_{1.0}\theta^1 + (L_1)_{0.1}\theta^2 - 5L_1(\varphi_1\theta^1 + \varphi_2\theta^2). \tag{29}$$

The pseudoinvariant V of the weight -3 is:

$$V = \nabla_\alpha L_1 = (L_1)_{1.0}B - (L_1)_{0.1}A - 5L_1(B\varphi_1 - A\varphi_2). \tag{30}$$

Pseudovectorial field ξ of the weight 3 is:

$$\xi = -2\Omega\alpha - \gamma, \tag{31}$$

in the case $A \neq 0$ field γ is:

$$\begin{aligned}\gamma^1 = &-\frac{6BN(BP + A_{1.0})}{5A^2} + \frac{18NBQ}{5A} + \\ &+\frac{6N(B_{1.0} + A_{0.1})}{5A} - N_{0.1} - \frac{12}{5}NR - 2\Omega B,\\ \gamma^2 = &-\frac{6N(BP + A_{1.0})}{5A} + N_{1.0} + \frac{6}{5}NQ + 2\Omega A,\end{aligned}$$

and in the case $B \neq 0$ is:

$$\begin{aligned}\gamma^1 = &-\frac{6N(AN - B_{0.1})}{5B} - N_{0.1} + \frac{6}{5}NR - 2\Omega B,\\ \gamma^2 = &-\frac{6AN(AS - B_{0.1})}{5B^2} + \frac{18NAR}{5B} - \\ &-\frac{6N(A_{0.1} + B_{1.0})}{5B} + N_{1.0} - \frac{12}{5}NQ + 2\Omega A.\end{aligned}$$

The pseudoinvariant Γ of the weight 4 is:

$$\begin{aligned}\Gamma = &\frac{\gamma^1\gamma^2(\gamma^1_{1.0} - \gamma^2_{0.1})}{M} + \frac{(\gamma^2)^2\gamma^1_{0.1} - (\gamma^1)^2\gamma^2_{1.0}}{M} + \\ &+\frac{P(\gamma^1)^3 + 3Q(\gamma^1)^2\gamma^2 + 3R\gamma^1(\gamma^2)^2 + S(\gamma^2)^3}{M}.\end{aligned} \tag{32}$$

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